Appendix

$$f(r,z)=\sum_{m=0}^{3}\sum_{n=0}^{3}a_{m,n}r^m z^n \ , \tag{1}$$

$$\begin{aligned}
&f_{k,l}=[f(r,z)]_{k,l}=\phi_{k,l} \ ,\\
&\left(\frac{\partial f}{\partial r}\right)_{k,l}=\left(\frac{\partial \phi}{\partial r}\right)_{k,l} \ ,\\
&\left(\frac{\partial f}{\partial z}\right)_{k,l}=\left(\frac{\partial \phi}{\partial z}\right)_{k,l} \ ,\\
&\left(\frac{\partial^2 f}{\partial r\partial z}\right)_{k,l}=\left(\frac{\partial^2 \phi}{\partial r\partial z}\right)_{k,l} \ ,
\end{aligned} \tag{2}$$

$$k=i,i+1 \ , \quad l=j,j+1 \ , \tag{3}$$

$$\boldsymbol{T}^T_{i,j}=\begin{pmatrix} z_{,\eta} & -z_{,\xi} \\ -r_{,\eta} & r_{,\xi}\end{pmatrix}_{i,j} \ , \qquad J_{i,j}=(r_{,\xi}z_{,\eta}-r_{,\eta}z_{,\xi})_{i,j} \ , \tag{4}$$

$$\left(\frac{\partial\phi}{\partial r} \ \frac{\partial\phi}{\partial z}\right)_{i,j}=J^{-1}_{i,j}\boldsymbol{T}^T_{i,j}\begin{pmatrix}\dfrac{\partial\phi}{\partial\xi}\\ \dfrac{\partial\phi}{\partial\eta}\end{pmatrix}_{i,j} \ , \tag{5}$$

$$\begin{aligned}
&\left(\frac{\partial\phi}{\partial\xi}\right)_{i,j}=(\phi_{i+1,j}-\phi_{i-1,j})/2 \ ,\\
&\left(\frac{\partial\phi}{\partial\eta}\right)_{i,j}=(\phi_{i,j+1}-\phi_{i,j-1})/2 \ ,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\frac{\partial^2\phi}{\partial r\partial z}&=(\phi_{,r})_{,\xi}\xi_{,z}+(\phi_{,r})_{,\eta}\eta_{,z}\\
&=\phi_{,\xi\xi}\xi_{,r}\xi_{,z}+\phi_{,\xi\eta}(\xi_{,z}\eta_{,r}+\xi_{,r}\eta_{,z})+\phi_{,\eta\eta}\eta_{,r}\eta_{,z} \ ,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\phi_{,\xi\xi}=\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j} \ ,\\
&\phi_{,\eta\eta}=\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1} \ ,\\
&\phi_{,\xi\eta}=(\phi_{i+1,j+1}-\phi_{i-1,j+1}-\phi_{i+1,j-1}+\phi_{i-1,j-1})/4 \ ,
\end{aligned} \tag{8}$$

$$f(\boldsymbol{y})=0 \ , \qquad \nabla f(\boldsymbol{y})\times(\boldsymbol{x}^0-\boldsymbol{y})=0 \ , \tag{9}$$

$$\begin{aligned}
&\boldsymbol{\delta}_1=-f(\boldsymbol{x}^k)\frac{\nabla f(\boldsymbol{x}^k)}{\nabla f(\boldsymbol{x}^k)\cdot\nabla f(\boldsymbol{x}^k)} \ ,\\
&\boldsymbol{x}^{k+1/2}=\boldsymbol{x}^k+\boldsymbol{\delta}_1 \ ,\\
&\boldsymbol{\delta}_2=(\boldsymbol{x}^0-\boldsymbol{x}^k)-\frac{(\boldsymbol{x}^0-\boldsymbol{x}^k)\cdot\nabla f(\boldsymbol{x}^k)}{\nabla f(\boldsymbol{x}^k)\cdot\nabla f(\boldsymbol{x}^k)}\nabla f(\boldsymbol{x}^k) \ ,\\
&\boldsymbol{x}^{k+1}=\boldsymbol{x}^{k+1/2}+\boldsymbol{\delta}_2 \ ,
\end{aligned} \tag{10}$$

where $k = 0, 1, 2, \cdots$.

$$|\delta_1|^2 + |\delta_2|^2 < 10^{-6}\Delta r \Delta z \ , \tag{11}$$

$$f(r,z) = \sum_{m=0}^{1}\sum_{n=0}^{1} a_{m,n} r^m z^n \ , \tag{12}$$

$$|\nabla\phi| = 1 \ , \tag{13}$$

$$\max\left(D_{ij}^{-x}\phi, -D_{ij}^{+x}\phi, 0\right)^2 + \max\left(D_{ij}^{-y}\phi, -D_{ij}^{+y}\phi, 0\right)^2 = 1 \ , \tag{14}$$

$$\begin{aligned} D_{ij}^{-x}\phi &= \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} \ , \\ D_{ij}^{+x}\phi &= \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} \ , \end{aligned} \tag{15}$$

$$(\boldsymbol{a}^T\boldsymbol{Q}\boldsymbol{a})\phi^2 + (2\boldsymbol{a}^T\boldsymbol{Q}\boldsymbol{b})\phi + (\boldsymbol{b}^T\boldsymbol{Q}\boldsymbol{b}) = 1 \ , \tag{16}$$

$$\begin{aligned} &\boldsymbol{Q} = (\boldsymbol{P}\boldsymbol{P}^T)^{-1} \ , \\ &\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \end{pmatrix} \ , \\ &\boldsymbol{P}_1 = (\boldsymbol{x} - \boldsymbol{x}_A)/|\boldsymbol{x} - \boldsymbol{x}_A| \ , \\ &\boldsymbol{P}_2 = (\boldsymbol{x} - \boldsymbol{x}_B)/|\boldsymbol{x} - \boldsymbol{x}_B| \ , \\ &\boldsymbol{a} = (a_1 \ \ a_2) = (\frac{1}{s_1} \ \ \frac{1}{s_2}) \ , \\ &\boldsymbol{b} = (b_1 \ \ b_2) = (-\frac{\phi_A}{s_1} \ \ -\frac{\phi_B}{s_2}) \ . \end{aligned} \tag{17}$$

$$\begin{aligned} (a_1\phi + b_1) &\geq (\boldsymbol{P}_1 \cdot \boldsymbol{P}_2^T)(a_2\phi + b_2) \ , \\ (a_2\phi + b_2) &\geq (\boldsymbol{P}_1 \cdot \boldsymbol{P}_2^T)(a_1\phi + b_1) \ , \end{aligned} \tag{18}$$

$$\boldsymbol{P}\nabla\phi = \boldsymbol{\nu} \ , \qquad \boldsymbol{\nu} = (\nu_1 \ \ \nu_2) \ , \tag{19}$$

$$\begin{aligned} &(\boldsymbol{P}^{-1}\boldsymbol{\nu}) \cdot (\boldsymbol{P}^{-1}\boldsymbol{\nu}) = 1 \ , \\ \text{or} \quad &(\boldsymbol{P}^{-1}\boldsymbol{\nu})^T \cdot (\boldsymbol{P}^{-1}\boldsymbol{\nu}) = 1 \ , \\ \text{or} \quad &\boldsymbol{\nu}^T(\boldsymbol{P}\boldsymbol{P}^T)^{-1}\boldsymbol{\nu} = 1 \ , \end{aligned} \tag{20}$$

$$\nu_1 = (\phi - \phi_A)/s_1 \ , \qquad \nu_2 = (\phi - \phi_B)/s_2 \ , \tag{21}$$

$$\Delta t < \min_{i,j}\left[\frac{\Delta r}{u}\ ,\frac{\Delta z}{v}\ ,\sqrt{We\frac{\rho_1+\rho_2}{8\pi}h^{3/2}}\ ,\frac{Re}{2}\frac{\rho^n}{\mu^n}\left(\frac{1}{\Delta r^2}+\frac{1}{\Delta z^2}\right)^{-1}\ ,\sqrt{\frac{2h}{|\boldsymbol{F}|}}\right] \ , \tag{22}$$

$$\Delta t_0 < \min_{i,j}\left[\sqrt{We\frac{\rho_1+\rho_2}{8\pi}h^{3/2}}\ ,\frac{Re}{2}\frac{\rho^n}{\mu^n}\left(\frac{1}{\Delta r^2}+\frac{1}{\Delta z^2}\right)^{-1}\right] \ , \tag{23}$$

$$\Delta t' < \min_{i,j} \left[\frac{\Delta r}{u} , \frac{\Delta z}{v} , \sqrt{\frac{2h}{|\boldsymbol{F}|}} \right] , \tag{24}$$

$$\Delta t = \min(\Delta t_0, \Delta t') , \tag{25}$$

$$\kappa(\phi) = J^{-1} \nabla_\Xi \cdot \left(g \boldsymbol{T} \frac{\boldsymbol{T}^T \nabla_\Xi \phi}{|\boldsymbol{T}^T \nabla_\Xi \phi|} \right) , \tag{26}$$

$$\nabla_\Xi \phi = \left(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2}, \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2} \right) , \tag{27}$$

$$\kappa_{critical} = 1.d0/\epsilon , \tag{28}$$

$$\kappa(\phi)_{i,j},\ \kappa(\phi)_{i+1,j},\ \kappa(\phi)_{i,j+1},\ \text{or } \kappa(\phi)_{i+1,j+1} > \kappa_{critical} , \tag{29}$$

$$\rho_1 = 1000\ Kg/m^3 , \qquad \mu_1 = 10. \times 10^{-3}\ Kg/m \cdot sec , \qquad \sigma = 0.032\ Kg/sec^2 , \tag{30}$$

$$Re = 4.8 , \qquad We = 9.6 \tag{31}$$

$$\rho_1 = 1070\ Kg/m^3 , \qquad \mu_1 = 3.5 \times 10^{-3}\ Kg/m \cdot sec , \qquad \sigma = 0.032\ Kg/sec^2 . \tag{32}$$